<u>Digression</u>: TQFTs in 2+1 d (MTCs)

A TQFT can be understood as a finite collection of anyons

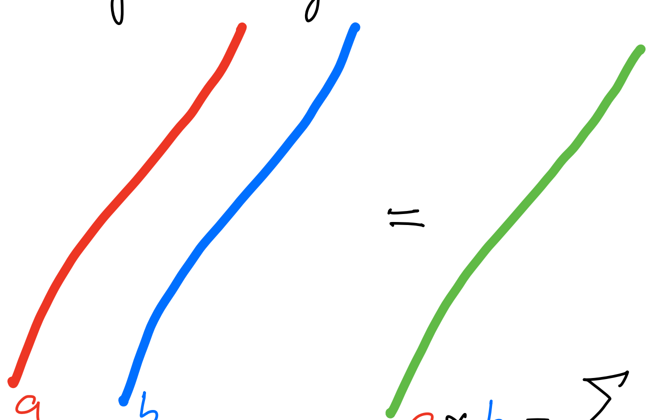
$\quad\quad\quad\quad$ || particles with fractional statistics

Define $\mathcal{A} := \{$ set of all Anyons $\}$ with the following additional data:

- <u>Fusion</u>: A commutative, associative product $\times : \mathcal{A} \times \mathcal{A} \longrightarrow \mathcal{A}$ given by

$$a \times b = \sum_{c \in \mathcal{A}} N_{ab}{}^{c}\, c$$

where $N_{ab}{}^{c} \in \mathbb{Z}_{\geq 0}$ are so-called "fusion coefficients". Denote the trivial anyon by $\mathbb{1}$.



$$= \quad a \times b = \sum_{c} N_{ab}{}^{c}\, c$$

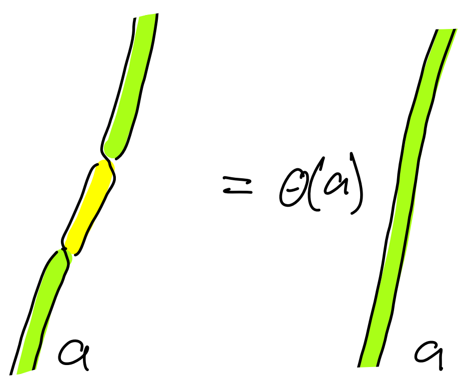
- <u>Topological spin:</u>

A map $\theta : \mathcal{A} \longrightarrow U(1)$ determining the anyonic character of an anyon.

Write $\quad \theta(a) := \exp(2\pi i h_a)$,

where $\quad h_a : \mathcal{A} \longrightarrow \mathbb{Q}/\mathbb{Z}$

is spin of a



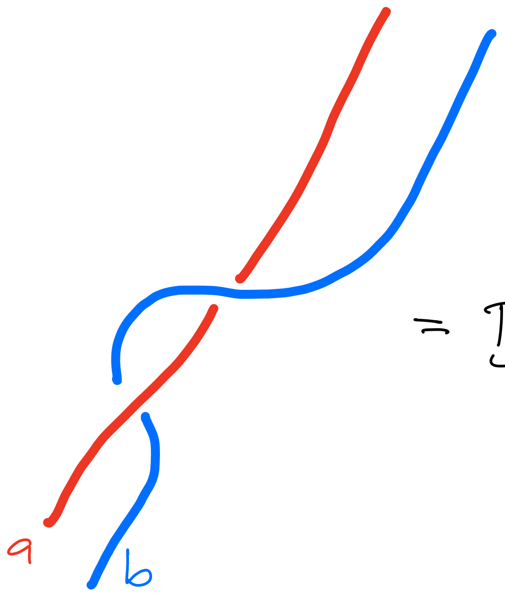
- <u>S- and T-matrices:</u>

A representation of the modular group

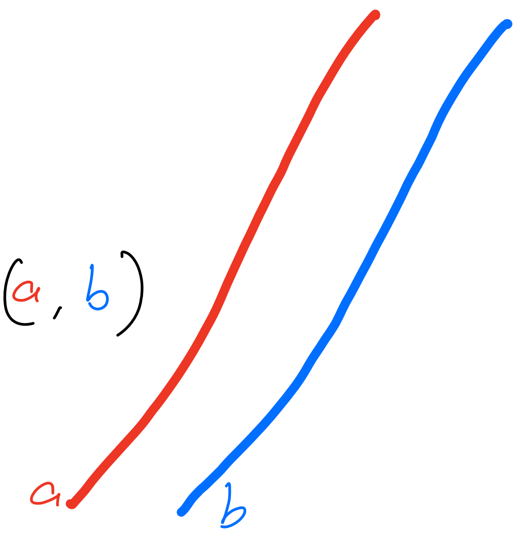
S-matrix determines the braiding phase $B : \mathcal{A} \times \mathcal{A} \longrightarrow U(1)$ between anyons

$$B(a,b) = \frac{S_{ab}}{S_{\mathbb{1}b}}$$

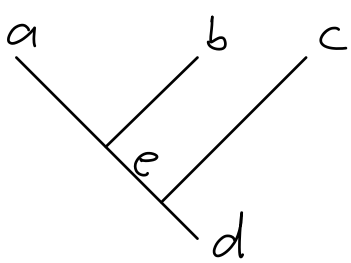
while $T_{ab} = \theta_a \, e^{-2\pi i c/24} \, \delta_{ab}$ where c is the "chiral central charge" of the TQFT



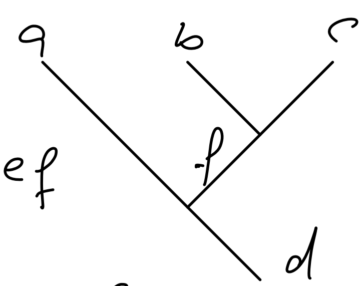
$= B(a, b)$

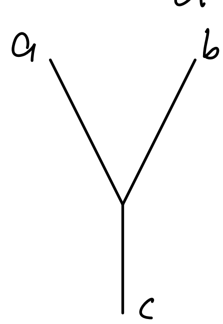


- <u>F- and R-symbols:</u>

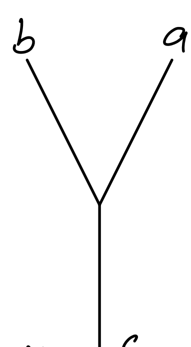


$= (F^{abcd})_{ef}$





$= R_c^{ab}$



with "Pentagon" and "Hexagon rels"

Abelian Anyons:

An anyon a is said to be "abelian" if the fusion of a with an arbitrary anyon b contains a single anyon $c = c(a,b)$

$$a \times b = c \quad \forall \; b \in \mathcal{A}$$

$$\Longleftrightarrow \quad \sum_{c \in \mathcal{A}} N_{ab}{}^{c} = 1 \quad \forall \; b \in \mathcal{A}$$

→ abelian a has unique inverse

$$a \times \bar{a} = \mathbb{1}$$

→ form finite abelian group

Abelian TQFTs:

An abelian TQFT is a TQFT in which all anyons in $\mathcal{A}$ are abelian

→ completely determined by the group $\mathcal{A}$ and the topological spin $\Theta : \mathcal{A} \longrightarrow U(1)$ (quadratic form on $\mathcal{A}$)

Then

$$\mathcal{B}(a,b) = \frac{\theta(a \times b)}{\theta(a)\,\theta(b)} \qquad a, b \in \mathcal{A}$$

$$S(a,b) = \frac{\mathcal{B}(a,b)}{\sqrt{|\mathcal{A}|}}\,, \quad T(a,b) = e^{-\frac{2\pi i c}{24}}\,\theta(a)\,\delta_{ab}$$

Any Abelian TQFT admits a rep. as an Abelian Chern-Simons theory

$$\longrightarrow \quad \mathcal{L} = \frac{1}{4\pi} a^t K a$$

for a $U(1)^n$ gauge field $a^t = (a_1, a_2, \dots, a_n)$

$\longrightarrow$ gauge invariant provided $K \in \mathbb{Z}^{n\times n}$ is symmetric and integral valued

The theory has central charge

$$c = \text{signature}(K)$$

Observables:

$$W_{\vec{\alpha}}(\gamma) := \exp\left[i \vec{\alpha}^t \oint_\gamma a \right] \quad \text{"Wilson line"}$$

where $\vec{\alpha} \in \mathbb{Z}^n$ is the representation

$$U(1)^n \ni \theta \longmapsto e^{i\vec{\alpha}\cdot\theta}$$

charge of $W_{\vec{\alpha}}$

$\rightarrow$ the $W_{\vec{\alpha}}(\gamma)$ are world-lines of Anyons with braiding

$$\mathcal{B}(\vec{\alpha},\vec{\beta}) := \exp\left[2\pi i\, \vec{\alpha}^{t} K^{-1} \vec{\beta}\right]$$

and top. spin

$$\theta(\vec{\alpha}) := \exp\left[2\pi i\, h_{\vec{\alpha}}\right], \quad h_{\vec{\alpha}} := \frac{1}{2}\vec{\alpha}^{t} K^{-1} \vec{\alpha}$$

θ is a "quadratic refinement" of the bilinear form $\mathcal{B}$:

$$\mathcal{B}(\vec{\alpha},\vec{\beta}) = \frac{\theta(\vec{\alpha}+\vec{\beta})}{\theta(\vec{\alpha})\,\theta(\vec{\beta})}$$

Bosonic and fermionic theories:

- if all diagonal components of K are even
  $\rightarrow$ all local operators are bosonic

→ Anyons are labelled by lattice points $\mathbb{Z}^n / K\mathbb{Z}^n$

→ $|\det K|$ independent anyons

- if at least one of the diagonal components of K is odd
  → theory contains local fermions
  → lines live in the lattice $(\mathbb{Z}^n / K\mathbb{Z}^n) \times \mathbb{Z}_2$
  → there are $2|\det K|$ indep. lines

Fusion rules: $\vec{\alpha} \times \vec{\beta} := (\vec{\alpha} + \vec{\beta} \mod K)$

The Abelian group $\mathcal{A}$ is given by

$$\bigoplus_{i=1}^{n} \mathbb{Z}_{k_i}$$

where k_i are obtained by bringing K into Smith normal form:

$$K \to \mathrm{diag}(k_1, k_2, \dots, k_n)$$

Back to $T[M_3; U(1)]$:

In last lecture we argued that

$T[M_3; U(1)]$ has a Lagrangian description given by:

$$\mathcal{L} = \frac{1}{4\pi} \int d^3x \; K^{ij} A_i \wedge dA_j$$

for M_3 given by surgery along link $L \subset S^3$ whose components L_i (with tubular neighborhood $S^1 \times D^2$) have linking matrix K^{ij}

$$\longrightarrow \; \underset{\#\{\text{vacua of } T[M_3; U(1)]\}}{\underset{\shortparallel}{|H_1(M_3;\mathbb{Z})|}} = |\det K|$$

To see this, note that we must have

$$\#\{\text{vacua of } T[M_3; U(1)]\}$$

3d-3d correspondence

$$\overset{!}{=} \#\{\text{vacua of } U(1)_{\mathbb{C}} \text{ CS-th on } M_3\}$$

$= \#\{ \text{flat } U(1)_{\mathbb{C}} \text{ connections on } M_3 \}$

$\longrightarrow$ parametrized by holonomies along 1-cycles

Know: $H_1(M_3; \mathbb{Z}) = \mathbb{Z}[\mu_1, \dots, \mu_n] / \operatorname{Im} K$

$\longrightarrow$ holonomies $x_i \in \mathbb{C}^*$ along 1-cycles μ_i are subject to the constraint

$$\prod_{i=1}^{n} x_i^{K_{ij}} = 1 \quad \forall \; j = 1, \dots, n$$

$\longrightarrow$ has exactly $|\det K|$ solutions!